RPERESENTATION THEORY BACKPAPER EXAMINATION

Attempt all questions. Total Marks: 100

(1) State and prove Maschke's Theorem and Schur's Lemma.
(5+5+10+10 = 30 marks)

(2) Let V and W be finite dimensional vector spaces over $\mathbb{C}$. Let G be a finite group.
  (a) Prove that $V^* \otimes W \cong Hom_{\mathbb{C}}(V, W)$ (isomorphism as vector spaces over $\mathbb{C}$, here V^* denotes the dual vector space $Hom_{\mathbb{C}}(V, \mathbb{C})$ of V).
  (b) Let V, W be G-modules. Prove that $V^* \otimes W$ and $Hom_{\mathbb{C}}(V, W)$ are both G-modules, and the above isomorphism is an isomorphism of G-modules.
(5+5 = 10 marks)

(3) (a) Write the character table for the dihedral group D_4 of order 8. Give details of all your calculations.
  (b) Let V denote the irreducible representation of D_4 of highest dimension (there exists a unique one). Compute the character of the second symmetric power $Sym^2(V)$ of V, and decompose $Sym^2(V)$ as a direct sum of irreducible representations of D_4.
(10+10 = 20 marks)

(4) (a) Write the character table for the symmetic group S_5 on 5 letters. Give details of all your calculations (the various conjugacy classes and their orders, the description of the characters, why each one is irreducible and how exactly you found the last 3 characters).
  (b) The same question for the alternating group A_5.
(20+20 = 40 marks)